



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 45

Option Pricing – American Options

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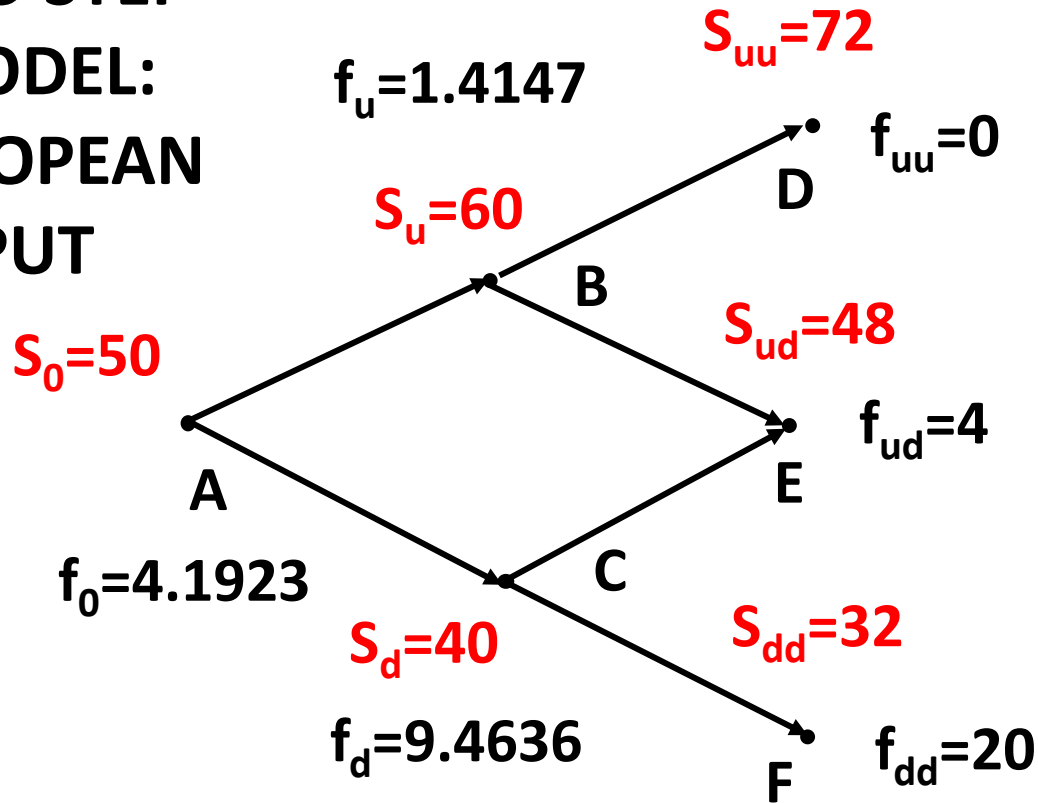
TWO PERIOD BINOMIAL MODEL

EXAMPLE 1

- Consider a 2-year European put with a strike price of 52 on a stock whose current price is 50. In each time step (of one year) the stock price either moves up by 20% or moves down by 20%. Let the risk-free interest rate be 5%. Calculate the current price of the option using a two step binomial model.



TWO STEP MODEL: EUROPEAN PUT



$u = 1.20; d = 0.80; T = 1 \text{ year}; r = 5\%$

Risk neutral probabilities

$$q_u = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \times 1} - 0.80}{1.20 - 0.80} = 0.6282$$

Stock prices at various nodes

$$S_{uu} = 72; S_{ud} = S_{du} = 48; S_{dd} = 32; S_u = 60; S_d = 40; S_0 = 50$$

Claim values at various nodes : $f(S_T) = \max(K - S_T, 0)$

$$f_{uu} = \max(52 - 72, 0) = 0; f_{ud} = \max(52 - 48, 0) = 4$$

$$f_{dd} = \max(52 - 32, 0) = 20$$

$$f_u = e^{-rT} [qf_{uu} + (1-q)f_{ud}]$$

$$= 0.9512 (0.6282 \times 0 + 0.3718 \times 4) = 1.4147$$

$$f_d = e^{-rT} [qf_{ud} + (1-q)f_{dd}]$$

$$= 0.9512 (0.6282 \times 4 + 0.3718 \times 20) = 9.4636$$

$$f_0 = e^{-rT} [qf_u + (1-q)f_d]$$

$$= 0.9512 (0.6282 \times 1.4147 + 0.3718 \times 9.4636)$$

$$= 4.1923$$

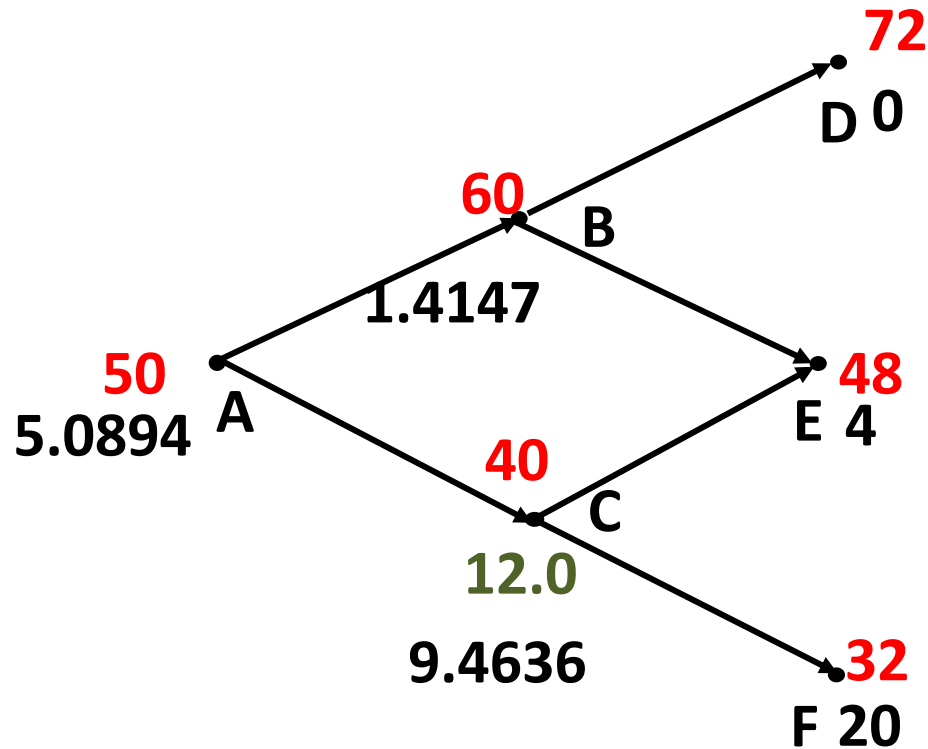
VALUING AN AMERICAN PUT OPTION

EXAMPLE 2

- Consider a 2-year American put with a strike price of 52 on a stock whose current price is \$50. In each time step (of one year) the stock price either moves up by 20% or moves down by 20%. Let the risk-free interest rate be 5%. Calculate the current price of the option using the two step binomial model.



TWO STEP MODEL: AMERICAN PUT



Stock prices at various nodes

$$S_{uu} = 72; S_{ud} = S_{du} = 48; S_{dd} = 32; S_u = 60; S_d = 40; S_0 = 50$$

Claim values at various end nodes :

$$f(S_T) = \max(K - S_T, 0)$$

$$f_{uu} = \max(52 - 72, 0) = 0;$$

$$f_{ud} = \max(52 - 48, 0) = 4$$

$$f_{dd} = \max(52 - 32, 0) = 20$$

$$\begin{aligned}
 f_u &= 1.4147; f_u^{\text{early exercise}} = \max(K - S_u, 0) = \max(52 - 60, 0) = 0 \\
 f_d &= 9.4636; f_d^{\text{early exercise}} = \max(K - S_d, 0) = \max(52 - 40, 0) = 12 \\
 f_0 &= 0.9512 \left(0.6282 \times 1.4147 + 0.3718 \overbrace{9.4636}^{12.0000} \right) = \overbrace{4.1923}^{5.0894} \\
 f_0^{\text{early exercise}} &= \max(K - S_0, 0) = \max(52 - 50, 0) = 2
 \end{aligned}$$

CONTINUOUS TIME (BLACK SCHOLES) MODEL



STEPS

- **BROWNIAN MOTION**
- **GENERALIZED BROWNIAN MOTION**
- **ITO EQUATION**
- **STOCK PRICE AS GENERALIZED BM**
- **FOKKER PLANCK EQUATION FOR STOCK PRICE**
- **SOLUTION OF FP EQUATION LOGNORMAL DISTRIBUTION OF STOCK PRICE**
- **BLACK SCHOLES PDE**
- **RISK NEUTRAL CALL VALUATION**



STOCHASTIC PROCESSES



PRICES OF STOCKS & OTHER FINANCIAL ASSETS



PRICES OF DERIVATIVE SECURITIES



TYPES OF EVOLUTION OF PROCESSES

- **DETERMINISTIC PROCESSES**
- **STOCHASTIC PROCESSES**
- **CHAOTIC PROCESSES**



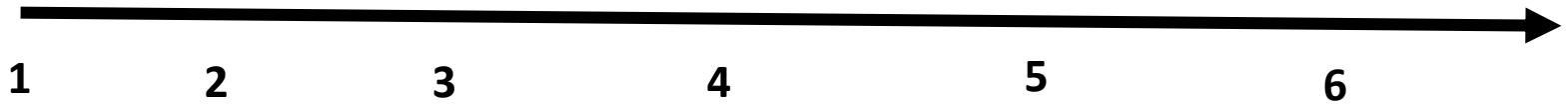
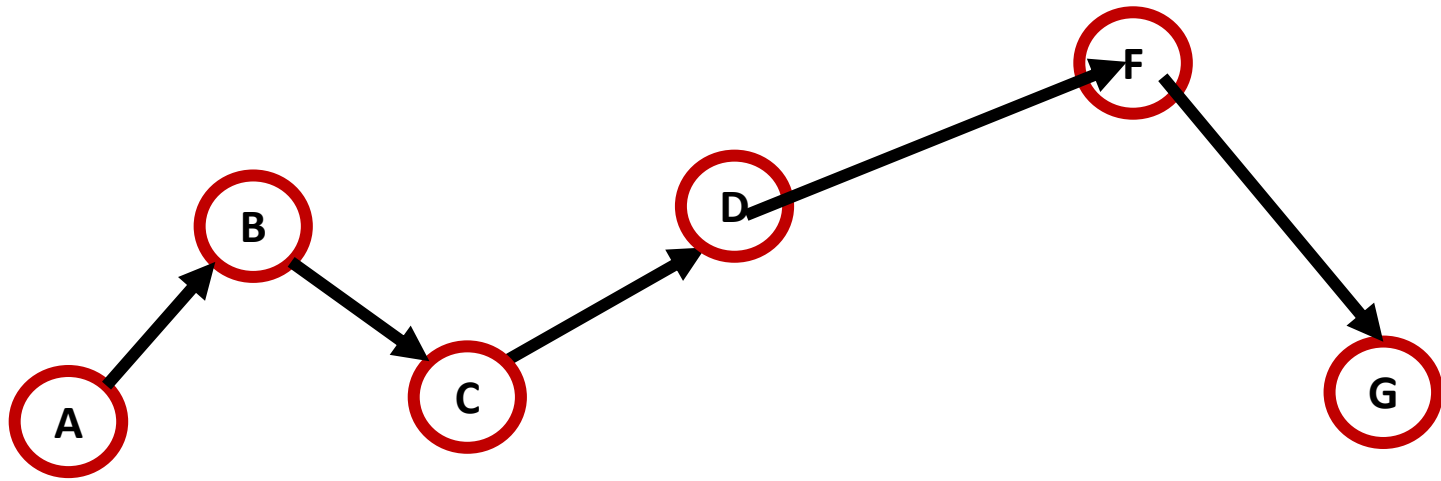
DETERMINISTIC PROCESSES

- Processes that evolve in time in a precisely predictable manner are called deterministic processes.
- In such processes, knowledge of the state of the system at any given point in time enables us to precisely predict its state at a future point in time.

- **A system will be deterministic if:**
- **We have full knowledge of the state of the system at a given point in time i.e. its state vector;**
- **We have knowledge of the evolutionary dynamics of the state vector in time.**

STOCHASTIC PROCESS

- A stochastic process is a process that evolves in time in a random manner. Thus, it can be represented by a sequence of random variables indexed with reference to the instants of time at which the process evolves.
- We can, thus, represent a stochastic process as a sequence $\{X_t; t \geq 0\}$ of random variables defined on a suitable probability space.



- We can represent a stochastic process as a sequence of random variables $\{X_t; t \in I\}$ defined on a suitable probability space. The *index set I , which usually represents time*, can either be discrete or continuous set.

DISCRETE TIME PROCESSES

- Stochastic processes usually evolve with time. They are, therefore, indexed with reference to points on the timeline.
- In discrete time processes, the process evolves (undergoes a transition) at discretely identified points on the timeline.
- Thus, from the perspective of the process, time is assumed to evolve in discontinuous jumps i.e. in discrete steps of a certain length.
- Discrete time can be represented by a lattice with lattice points labelled by integers.

TIME LINE VS TIME STEPS

- The time steps are those discrete points on the timeline, usually spaced equidistant apart, at which the state of the observed system undergoes a transition.
- Thus, from the system's perspective time evolves in discontinuous jumps.

DISCRETE VARIABLE PROCESSES

- In discrete variable processes, the random variables comprising the stochastic process map to a discrete set i.e. the random variables forming the stochastic process are discrete.
- Discrete variables can be represented by a lattice with lattice points labelled by integers.

MARKOV PROCESS

- For a Markov process, we can say that the future state of the process is dependent only on its present state and not on the history of how the process reached the present state.



- **Its memory is restricted, at any instant of time, to the immediately preceding time argument alone.**
- **That is, the current evolution of a Markov process depends only on its immediately preceding state and on no earlier states.**